Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(00)00090-3
J. Appl. Maths Mechs, Vol. 64, No. 4, pp. 635–645, 2000

0021–8928/00/$—see front matter

APPLICATION OF FUNCTIONS OF A COMPLEX VARIABLE TO CERTAIN THREE-DIMENSIONAL PROBLEMS OF ELASTICITY THEORY†

G. Z. SHARAFUTDINOV

Moscow

e-mail: common@inimech.msu.ru

(*Received* 30 *March* 2000)

Methods of the theory of functions of a complex variable are applied to problems of the deformation of thin plates of constant or variable thickness considered in three dimensions. To that end, a third complex potential is added to the two complex Kolosov–Muskhelishvili potentials. The components of the displacement vector and the stress tensor are represented in terms of these three complex potentials. The formulations if the problems, characteristic for cases in which complex variables are used in problems of elasticity theory, are investigated.

The theory of functions of a complex variable is of considerable value in solving two-dimensional problems of elasticity theory [1]. However, problems concerning the deformation of thin plates of variable thickness have hitherto remained outside its sphere of applicability. It is also noteworthy that the system of equations for the problem of a plane stressed state need not necessarily satisfy all the compatibility conditions, and so solutions of problems of a plane stressed state are generally approximate [2]. Finally, in some problems, such as those related to the study of the stress–strain state in the neighbourhood of a crack tip, even in plates of constant thickness, it seems preferable to take the three-dimensional nature of the stress and strain distribution into consideration. Obviously, the solution of these problems is possible only when they are considered in the context of three-dimensional elasticity theory. We will do this, proceeding from the general representation of the solution for a thin plate of variable thickness.

1. THE STRESS-STRAIN IN A THIN PLATE OF VARIABLE THICKNESS

Let us assume that a thin elastic plate of variable thickness is deformed by forces uniformly distributed over its thickness. We introduce a rectangular cartesian system of coordinates $OX_1X_2X_3$ in such a way that the coordinate plane X_1OX_2 coincides with the middle plane of the plate and the coordinate axis OX_3 is perpendicular to it. We will assume that the plate is symmetrical about the middle plane and that its boundary surfaces are described by the equations $x_3 = \pm h(x_1, x_2)$. The normal to the boundary surface of such a plate will not necessarily coincide with the normal to the middle plane, so that the conditions for a plane stressed state are violated [3]. For that reason, we will proceed from the general equations of elasticity theory, assuming, however, that the plate is thin.

Let us assume that the components u_1 and u_2 of the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ are independent of the third coordinate x_3, but $u_3 = u_3(x_1, x_2, x_3)$. Since under these conditions of deformation the plate will remain symmetrical about the middle plane, we conclude that u_3 is an odd function of x_3 and therefore, assuming the plate is thin, we can confine our attention to the first term of the expansion, writing.

$$u_3(x_1, x_2, x_3) = g(x_1, x_2)x_3$$

This implies the following expressions for the components of the strain tensor

†*Prikl. Mat. Mekh.* Vol. 64, No. 4, pp. 659–669, 2000.

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1},\quad \varepsilon_{22}=\frac{\partial u_2}{\partial x_2},\quad \varepsilon_{33}=g(x_1,\ x_2)$$

$$\varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right),\quad \varepsilon_{13}=\frac{1}{2}\frac{\partial g}{\partial x_1}x_3,\quad \varepsilon_{23}=\frac{1}{2}\frac{\partial g}{\partial x_2}x_3$$

The components of the stress tensor for an elastic material may be written in the form

$$\sigma_{11}=\lambda\theta+2\mu\frac{\partial u_1}{\partial x_1},\quad \sigma_{22}=\lambda\theta+2\mu\frac{\partial u_2}{\partial x_2},\quad \sigma_{33}=\lambda\theta+2\mu g(x_1,\ x_2)$$

$$\sigma_{12}=\mu\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right),\quad \sigma_{13}=\mu\frac{\partial g}{\partial x_1}x_3,\quad \sigma_{23}=\mu\frac{\partial g}{\partial x_2}x_3 \tag{1.1}$$

$$\theta=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+g$$

where λ and μ are the Lamé constants.

In the general case, the components of the stress tensor (1.1), which define the stressed state of the deformed body, must satisfy the equilibrium equations

$$\sigma_{ij,j}=0$$

and certain compatibility conditions, which we take to be the Beltrami–Michell equations [3].

Taking the above assumptions into account, we write the equations of equilibrium in the form

$$\frac{\partial}{\partial x_1}(\sigma_{11}+\mu g)+\frac{\partial\sigma_{12}}{\partial x_2}=0,\quad \frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial}{\partial x_2}(\sigma_{22}+\mu g)=0,\quad \frac{\partial^2 g}{\partial x_1^2}+\frac{\partial^2 g}{\partial x_2^2}=0 \tag{1.2}$$

The third equation shows that $g(x_1, x_2)$ is a harmonic function.

The sum of the components $\sigma_{11}+\sigma_{22}$ of the stress tensor obviously satisfies the equation

$$\nabla^2(\sigma_{11}+\sigma_{22})=0,\quad \nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2} \tag{1.3}$$

Indeed, we substitute into the compatibility condition

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}=2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2} \tag{1.4}$$

the Hooke's law, relations

$$\varepsilon_{11}=\frac{1}{E}[\sigma_{11}-\nu(\sigma_{22}+\sigma_{33})],\quad \varepsilon_{22}=\frac{1}{E}[\sigma_{22}-\nu(\sigma_{11}+\sigma_{33})],\quad \varepsilon_{12}=\frac{1+\nu}{E}\sigma_{12}$$

Taking into consideration the relation

$$\frac{\partial^2\sigma_{12}}{\partial x_1\partial x_2}=-\frac{\partial^2\sigma_{11}}{\partial x_1^2}-\frac{\partial^2\sigma_{22}}{\partial x_2^2}-\mu\nabla^2 g=-\frac{\partial^2\sigma_{11}}{\partial x_1^2}-\frac{\partial^2\sigma_{22}}{\partial x_2^2}$$

which follows from the first two of equilibrium equations (1.2), we deduce from compatibility condition (1.4) that

$$\nabla^2(\sigma_{11}+\sigma_{22})-\nu\nabla^2\sigma_{33}=0 \tag{1.5}$$

It follows from Hooke's law that

$$\sigma_{33}=E\varepsilon_{33}+\nu(\sigma_{11}+\sigma_{22})=Eg+\nu(\sigma_{11}+\sigma_{22})$$

Taking the third equilibrium equation (1.2) into account, we obtain

$$\nabla^2\sigma_{33} = \nu\nabla^2(\sigma_{11} + \sigma_{22})$$

The validity of Eq. (1.3) follows from this equality and from (1.5).

We now introduce the stress function $\Phi(x_1, x_2)$:

$$\sigma_{11} + \mu g = \frac{\partial^2\Phi}{\partial x_2^2}, \quad \sigma_{22} + \mu g = \frac{\partial^2\Phi}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2\Phi}{\partial x_1 \partial x_2}$$

This function obviously satisfies the first two equilibrium equations (1.2), and moreover

$$\sigma_{11} = \frac{\partial^2\Phi}{\partial x_2^2} - \mu g, \quad \sigma_{22} = \frac{\partial^2\Phi}{\partial x_1^2} - \mu g$$

Substituting these expressions into Eq. (1.3) and using the fact that the function $g(x_1, x_2)$ is harmonic, we get

$$\nabla^2(\nabla^2\Phi - 2\mu g) = \nabla^4\Phi - 2\mu\nabla^2 g = \nabla^4\Phi = 0$$

that is, the stress function is a biharmonic function. We may therefore apply the methods of the theory of functions of a complex variable.

2. THE COMPLEX REPRESENTATION OF THE COMPONENTS OF THE DISPLACEMENT VECTOR AND THE STRESS AND STRAIN TENSORS

It is well known (see, for example [1]) that any biharmonic function may be expressed using Goursat's formula as functions of complex variables $z = x_1 + ix_2$; $\bar{z} = x_1 - ix_2$:

$$2\Phi(z, \bar{z}) = \bar{z}\varphi(z) + z\bar{\varphi}(\bar{z}) + \chi(z) + \bar{\chi}(\bar{z}) \tag{2.1}$$

In the plane theory of elasticity, we have the following representations for combinations of the components of the stress tensor [1]

$$\sigma_{11} + \sigma_{22} = 2S_1, \quad S_1 = [\varphi'(z) + \bar{\varphi}'(\bar{z})] \tag{2.2}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2S_2, \quad S_2 = [\bar{z}\varphi''(z) + \psi'(z)] \tag{2.3}$$

where $\psi(z) = \chi'(z)$.

However, the problem under consideration, concerning the deformation of a thin plate of variable thickness, cannot generally be considered in the context of two-dimensional problems of elasticity theory. Therefore, the complex representations for the components of the displacement vector and stress and strain tensors will undergo certain changes.

It follows directly from the representation of the stress function that

$$\sigma_{11} + \sigma_{22} + 2\mu g = 4\frac{\partial^2\Phi}{\partial z \partial \bar{z}} \tag{2.4}$$

$$\sigma_{11} - \sigma_{22} + 2i\sigma_{12} = -4\frac{\partial^2\Phi}{\partial \bar{z}^2} \tag{2.5}$$

Using Goursat's formula (2.1), we thus deduce from (2.4) and (2.5) that

$$\sigma_{11} + \sigma_{22} + 2\mu g = 2S_1 \tag{2.6}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2S_2 \tag{2.7}$$

The last relations express combinations of components of the stress tensor in terms of two functions of a complex variable. However, unlike (2.2) and (2.3), these relations also involve a component of the strain tensor, $\varepsilon_{33} = g(x_1, x_2)$. Let us express this quantity, as well as the components of the displacement vector u_1 and u_2, in complex form.

We introduce a complex displacement by

$$D = u_1 + iu_2$$

It can be shown that

$$2\frac{\partial D}{\partial \bar{z}} = \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} + i\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = \varepsilon_{11} - \varepsilon_{22} + 2i\varepsilon_{12} \tag{2.8}$$

Using Hooke's law, we obtain from (2.5)

$$4\mu\frac{\partial D}{\partial \bar{z}} = -4\frac{\partial^2 \Phi}{\partial \bar{z}^2}$$

This equation will obviously be satisfied if we put

$$4\mu D(z, \bar{z}) = f(z) - 4\frac{\partial \Phi(z, \bar{z})}{\partial \bar{z}}$$

or, using Goursat's formula (2.1),

$$4\mu D(z, \bar{z}) = 4\mu(u_1 + iu_2) = f(z) - 2[\varphi(z) + z\bar{\varphi}'(\bar{z}) + \bar{\psi}(\bar{z})] \tag{2.9}$$

This expression is an analogue of Kolosov's formula for displacement, as may readily be verified by reducing it to that formula:

$$2\mu(u_1 + iu_2) = \varkappa\varphi(z) - z\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z})$$

$$\kappa = \begin{cases} 3 - 4\nu \text{ in the case of plane strain,} \\ (3 - \nu)/(1 + \nu) \text{ in the case of a plane stressed state} \end{cases}$$

Indeed, differentiating Eq. (2.9) with respect to z and then taking real parts we obtain

$$\varepsilon_{11} + \varepsilon_{22} = \frac{1}{\mu}(S_3 - S_1), \quad S_3 = \frac{1}{4}[f'(z) + \bar{f}'(\bar{z})] \tag{2.10}$$

In the case of a plane strain, it follows from Hooke's law that

$$\varepsilon_{11} + \varepsilon_{22} = \frac{\sigma_{11} + \sigma_{22}}{2(\lambda + \mu)}$$

Taking (2.2) into account, we substitute this relation into (2.10). The result is

$$f'(z) + \bar{f}'(\bar{z}) = \frac{4(\lambda + 2\mu)}{\lambda + \mu}S_1 = 8(1 - \nu)S_1$$

Thus, taking the relation between the functions $f(z)$ and $\varphi(z)$ as in the form

$$f(z) = 8(1 - \nu)\varphi(z)$$

we arrive at Kolosov's formula for the case of plane strain

Similar arguments show that formula (2.9) can be reduced to Kolosov's formula for a two-dimensional stressed state. In that case the relation between the functions $f(z)$ and $\varphi(z)$ is taken in the form

$$f(z) = \frac{8}{1 + \nu}\varphi(z)$$

To express the strain ε_{33} (that is, the function g) in complex form, we consider the relations of Hooke's law

$$\sigma_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{11}$$
$$\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{22}$$

Adding these equalities together and adding $2\mu g$ to the left- and right-hand sides of the resulting relation, we get

$$g = \frac{1}{2(\lambda+\mu)}(\sigma_{11} + \sigma_{22} + 2\mu g) - (\varepsilon_{11} + \varepsilon_{22})$$

and, in view of (2.6) and (2.10), we finally obtain

$$g = \frac{\lambda+2\mu}{\mu(\lambda+\mu)} S_1 - \frac{1}{\mu} S_3 \tag{2.11}$$

This representation now enables us to modernize relation (2.6), eliminating the function g from it

$$\sigma_{11} + \sigma_{22} = -\frac{2\mu}{\lambda+\mu} S_1 + 2S_3 \tag{2.12}$$

Thus, the real and imaginary parts of (2.7) and (2.9), together with relations (2.11) and (2.12), yield six equations for determining six quantities: three components of the stress tensor σ_{11}, σ_{22}, σ_{12}, two components of the displacement vector u_1, u_2, and the value of the third strain $\varepsilon_{33} = g$.

However, the stress–strain state is determined not only by these quantities but also by the components σ_{33}, σ_{13} and σ_{23} of the stress tensor (and of the strain tensor, ε_{33}, ε_{13} and ε_{23}). It should also be borne in mind that these quantities may be used in the boundary conditions. For example, when the boundary conditions at the boundary surface of the plate are prescribed, the components of the stress tensor indicated above will be involved to some degree or another. Hence, they must be expressed in terms of the functions of a complex variable introduced previously.

Omitting the straightforward calculations, we present the formulae for the components of the stress tensor in terms of the functions of a complex variable

$$\begin{aligned}
&\sigma_{11} = S_3 - \frac{\mu}{\lambda+\mu} S_1 - \operatorname{Re} S_2, \quad \sigma_{22} = S_3 - \frac{\mu}{\lambda+\mu} S_1 + \operatorname{Re} S_2\\
&\sigma_{33} = \frac{3\lambda+4\mu}{\lambda+\mu} S_1 - 2S_3, \quad \sigma_{12} = \operatorname{Im} S_2\\
&\sigma_{13} = (T_1 - T_3)x_3, \quad T_1 = \frac{\lambda+2\mu}{\lambda+\mu}[\varphi''(z) + \overline{\varphi}''(\bar{z})], \quad T_3 = \frac{1}{4}[f''(z) + \bar{f}''(\bar{z})]\\
&\sigma_{23} = i(T_2 - T_4)x_3, \quad T_2 = \frac{\lambda+2\mu}{\lambda+\mu}[\varphi''(z) - \overline{\varphi}''(\bar{z})], \quad T_4 = \frac{1}{4}[f''(z) - \bar{f}''(\bar{z})]
\end{aligned} \tag{2.13}$$

It can be shown that these expressions for the components of the stress tensor in terms of three functions of a complex variable satisfy the Beltrami–Michell compatibility conditions, and since (by construction) they satisfy the equations of equilibrium, they may be used to solve problems of elasticity theory of the class under consideration.

3. BOUNDARY CONDITIONS

It is well known that the boundary conditions in problems of elasticity theory are determined either by given external distributed forces $\mathbf{F} = (f_1, f_2, f_3)$, by given displacements of points on the body surface, or by external surface forces specified on one part of the surface and displacements on the other. Boundary conditions of the first type are expressed by the equalities

$$\sigma_{ij} n_j = f_i$$

where n_j are the components of the unit vector along the outward normal to the surface (to the contour) at the point under consideration. Boundary conditions of the second type are expressed by the equalities

$$u_i = q_i$$

where q_i are given functions on the surface (the contour).

To solve problems of the deformation of thin plates of variable thickness, one must thus determine the three functions of a complex variable, $\varphi(x)$, $f(x)$ and $\psi(x)$. These functions must satisfy certain conditions at the boundaries of the body being deformed. Thus, the need arises to formulate boundary conditions with reference to these complex functions.

Boundary conditions of the first type are known to reflect the equilibrium of a certain elementary volume of the deformed body. As applied to the case of thin plates of variable thickness, this volume will be a certain triangle of variable thickness (in the case when distributed stresses are specified on the contour, e.g. holes in the plate). The other version of the boundary conditions of a similar type, which differs somewhat from the first, is imposed on the plate surface.

We will first consider the case when the boundary conditions are specified on the contour of a cut in the plate. Let us assume that ds is an arc element on the contour of the boundary of the deformed body. The components of the unit vector of the outward normal $\mathbf{n} = (n_1, n_2, n_3)$ represent the cosines of the angles between this vector and the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ and are equal, respectively, to [1]

$$n_1 = \cos(\mathbf{n},\ \mathbf{e}_1) = \frac{dx_2}{ds},\quad n_2 = \cos(\mathbf{n},\ \mathbf{e}_2) = -\frac{dx_1}{ds},\quad n_3 = \cos(\mathbf{n},\ \mathbf{e}_3) = 0$$

Since the plate thickness is variable, we single out a layer of thickness dx_3. Then, for unit length of the contour we have

$$\sigma_{ij} n_j dx_3 = f_i dx_3$$

Integrating these expressions with respect to x_3, we get

$$\int\limits_{-h(x_1,x_2)}^{h(x_1,x_2)} \sigma_{ij} n_j dx_3 = 2hf_i$$

When the load is symmetrical about the middle plane, the resultant is $f_3 = 0$ and, since σ_{31} and σ_{32} are linear functions of x_3, the third relation becomes an identity $0 \equiv 0$. Thus, for the class of problems under consideration, it is meaningful to formulate the boundary conditions (along the contours of holes in the plate or the contours bounding the body) using only the first two equations; taking into consideration that σ_{31} and σ_{32} depend linearly on x_3, we take these equations in the form

$$\sigma_{kl} n_l = f_k(x_1, x_2),\quad k, l = 1, 2$$

Using the stress function to represent the components of the stress tensor, and the differential relations determined above for the components of the direction vector, we have

$$\left(\frac{\partial^2\Phi}{\partial x_2^2} - \mu g\right)\frac{dx_2}{ds} + \frac{\partial^2\Phi}{\partial x_1 \partial x_2}\frac{dx_1}{ds} = f_1$$

$$-\frac{\partial^2\Phi}{\partial x_1 \partial x_2}\frac{dx_2}{ds} - \left(\frac{\partial^2\Phi}{\partial x_1^2} - \mu g\right)\frac{dx_1}{ds} = f_2$$

These relations may be reduced to the form

$$d\frac{\partial\Phi}{\partial x_2} - \mu g dx_2 = f_1 ds,\quad -d\frac{\partial\Phi}{\partial x_1} + \mu g dx_1 = f_2 ds \tag{3.1}$$

We form a complex expression

$$(f_1 + if_2)ds = -id\left(\frac{\partial\Phi}{\partial x_1} + i\frac{\partial\Phi}{\partial x_2}\right) + i\mu g(dx_1 + idx_2)ds$$

Using the expression for the stress function, we obtain

$$d[\varphi(z) + z\overline{\varphi}'(\bar{z}) + \overline{\varphi}(\bar{z})] - \mu g dz = i(f_1 + if_2)ds$$

Since in this case the differential has the form

$$d = \frac{\partial}{\partial z}dz + \frac{\partial}{\partial \bar{z}}d\bar{z}$$

it follows from the last relation that

$$[\varphi'(z)dz + \overline{\varphi}'(\bar{z})dz + z\overline{\varphi}''(\bar{z})d\bar{z} + \overline{\psi}'(\bar{z})d\bar{z}] - \mu g dz = i(f_1 + if_2)ds$$

We now use formula (2.11); we obtain

$$S_3 dz - \frac{\mu}{\lambda+\mu}S_1 dz + \bar{S}_2 dz = i(f_1 + if_2)ds \tag{3.2}$$

On the contour of a circular hole of radius a, we have

$$z = ae^{i\theta}, \quad dz = iae^{i\theta}d\theta, \quad d\bar{z} = -iae^{-i\theta}d\theta$$

As a result, Eq. (3.2) becomes

$$a\left[\left(S_3 - \frac{\mu}{\lambda+\mu}S_1\right)e^{i\theta} - \bar{S}_2 e^{-i\theta}\right]d\theta = i(f_1 + if_2)ds$$

But if there are no stresses on the circular contour, the expression in square brackets will vanish. Finally, changing to conjugates, we obtain the following well-known expression [1]

$$S_3 - \frac{\mu}{\lambda+\mu}S_1 - S_2 e^{2i\theta} = 0$$

which is very convenient for prescribing boundary conditions for the stresses on the contour of a circular hole.

We will now express the boundary conditions for the stresses in the plate surface in terms of the functions of a complex variable introduced previously. Since we are considering plates which are both symmetrical about the middle surface and symmetrically loaded about it, we will confine attention to only one side of the plate.

Lets us assume that the form of the plate surfaces is defined by the equations

$$x_3 = \pm h(z, \bar{z})$$

The components l_1, l_2 and l_3 of the vector $\mathbf{N} = (l_1, l_2, l_3)$ normal to the plate surface have the form

$$l_1 = \frac{h_1 + h_2}{A}, \quad l_2 = i\frac{h_1 - h_2}{A}, \quad l_3 = -\frac{1}{A}$$

$$h_1 = \frac{\partial h}{\partial z}, \quad h_2 = \frac{\partial h}{\partial \bar{z}}, \quad A = \sqrt{1 + 4h_1 h_2}$$

The boundary conditions of the first type on the plate surface have the form

$$\sigma_{ij} l_j = p_i$$

Using the representations derived above for the components of the stress tensor, we obtain at $x_3 = h$

$$h_1\left[S_3-\frac{\mu}{\lambda+\mu}S_1-\bar{S}_2\right]+h_2\left[S_3-\frac{\mu}{\lambda+\mu}S_1-S_2\right]+h(T_3-T_1)=Ap_1$$

$$h_1\left[S_3-\frac{\mu}{\lambda+\mu}S_1+\bar{S}_2\right]-h_2\left[S_3-\frac{\mu}{\lambda+\mu}S_1+S_2\right]+h(T_4-T_2)=Ap_2$$

$$2h\frac{\lambda+2\mu}{\lambda+\mu}[h_2\varphi''(z)+h_1\bar{\varphi}''(\bar{z})]+\frac{1}{2}h[h_2 f''(z)+h_1\bar{f}''(\bar{z})]-$$

$$-\frac{3\lambda+4\mu}{\lambda+\mu}S_1+\frac{1}{2}S_3=Ap_3$$

Boundary conditions for the displacements are formulated on the basis of representation (2.9) for the displacement vector in complex form

$$4\mu(u_1+iu_2)=f(z)-2[\varphi(z)+z\bar{\varphi}'(\bar{z})+\bar{\psi}(\bar{z})]=4\mu q(z,\bar{z})$$

and displacements u_3 of the plate surfaces are prescribed using Eq. (2.11)

$$u_3=g(z,\bar{z})h(z,\bar{z})=\left(\frac{\lambda+2\mu}{\mu(\lambda+\mu)}S_1-\frac{1}{\mu}S_3\right)h(z,\bar{z})=Q(z,\bar{z})$$

4. ON THE UNIQUENESS OF THE COMPLEX POTENTIALS

The question of the degree to which the complex potentials introduced above are uniquely defined is solved essentially as in [1], except that here we are concerned with three potentials. The components σ_{11}, σ_{22} and σ_{12} of the stress tensor are determined using relations (2.7) and (2.12). In this case, besides these components of the stress tensor, some of the other components among σ_{33}, σ_{13} and σ_{23} must also be specified. Let us assume that the specified stress is σ_{33}, as given by the third relation of (2.13). Given these components of the stress tensor, relation (2.12) and the third relation of (2.13) form a system of equations in the real parts of the functions $\varphi'(z)$ and $f'(z)$. Solving these equations for the combinations S_1 and S_3, we obtain

$$S_1=\frac{\lambda+\mu}{3\lambda+2\mu}[\sigma_{11}+\sigma_{22}+\sigma_{33}]$$

$$S_3=\frac{1}{2(3\lambda+2\mu)}[(3\lambda+4\mu)(\sigma_{11}+\sigma_{22})+2\mu\sigma_{33}]$$

Hence it follows that, for given stresses, the real parts of the functions $\varphi'(z)$ and $f'(z)$ are uniquely defined, but their imaginary parts are only defined apart from the pure imaginary constants iA_0^*, iC_0^*, where A_0^*, C_0^* are real constants. Since

$$\varphi(z)=\int\varphi'(z)dz,\quad f(z)=\int f'(z)dz$$

it is obvious that adding expressions of the form $iA_0^*z+\alpha_0$, $iC_0^*z+\gamma_0$ to the functions $\varphi(z)$ does not change the stressed state of the deformed body.

Proceeding further from relations (2.7)

$$\sigma_{22}-\sigma_{11}+2i\sigma_{12}=2[\bar{z}\varphi''(z)+\psi'(z)]$$

we can show [1] that $\psi'(z)$ is uniquely defined, while $\psi(z)$ is defined apart from the complex constant β_0.

For given stresses, one can choose constants [1]

$$A_0^*,\ C_0^*,\ \alpha_0,\ \gamma_0,\ \beta_0 \tag{4.1}$$

so that the following equalities hold

$$\varphi(0)=0,\ \operatorname{Im}(\varphi'(0))=0,\ f(0)=0,\ \operatorname{Im}(f'(0))=0,\ \psi(0)=0$$

– this essentially exhausts the arbitrariness in the choice of the constants.

If the components of the displacement vector are given, the components of the stress tensor are uniquely defined, whence we conclude that in this case too the functions $\varphi(z)$, $\psi(z)$ and $f(z)$ must be defined with the accuracy indicated above. It follows directly from (21.9) that in that case the increment $\Delta(u_1+iu_2)$ to the components of the displacement vector is defined apart from

$$4\mu\Delta(u_1+iu_2)=iC_0^*z+(\gamma_0-2\alpha_0-2\bar{\beta}_0)$$

Hence we conclude that the displacements will be uniquely defined provided that

$$C_0^*=0,\quad \gamma_0-2\alpha_0-2\bar{\beta}_0=0 \tag{4.2}$$

Thus, for given displacements, the degree of arbitrariness in the choice of constants (4.1) is determined by (4.2). We will now consider the possibility of choosing the quantities

$$\varphi(0)=0\ \ (\text{or }\psi(0)=0),\ f(0)=0$$

thus eliminating the arbitrary element in the choice of the complex constants and ensuring that the displacements are uniquely specified [1].

Despite the fact that the components of the displacement vector and the stress tensor are single-valued functions, the functions of the complex variable $\varphi(z)$, $\psi(z)$ and $f(z)$ may turn out to be multivalued.

Indeed, these holomorphic functions will be single-valued in any simply connected domain occupied by the body. Let us assume now, following the approach in [1], that the domain is multiply connected, that is, the domain occupied by the body is bounded by simple closed contours $L_1, L_2, \dots, L_m$, L_{m+1}, the last of which contains all the others in its interior. It is assumed that these contours do not intersect.

It was established above that the real parts of the functions $\varphi'(z)$ and $f'(z)$ are single-valued. However, a circuit around any closed contour L_k' enclosing L_k will cause the imaginary parts of these functions to receive increments of the form $2\pi iA_k^*$ and $2\pi iC_k^*$, where A_k^*, C_k^* are real constants. Such an increment is guaranteed by the function $\ln z$, and we shall therefore assume that

$$\varphi'(z)=\sum_{k=1}^{m}A_k^*\ln(z-z_k)+F_\varphi,\quad f'(z)=\sum_{k=1}^{m}C_k^*\ln(z-z_k)+F_f \tag{4.3}$$

where m is the number of contours forming the boundary of the body occupying a certain domain; F_φ and F_f are holomorphic functions in that domain; z_k is a fixed point, arbitrarily chosen within the contour L_k.

Integrating relations (4.3), we obtain (summation will henceforth be carried out from $k=1$ to $k=m$)

$$\begin{aligned}&\varphi(z)=\int_{z_0}^{z}\varphi'(z)dz+\text{const}=\sum A_k^*\Gamma_k+I_\varphi+\text{const}\\&f(z)=\int_{z_0}^{z}f'(z)dz+\text{const}=\sum C_k^*\Gamma_k+I_f+\text{const}\\&\Gamma_k=(z-z_k)\ln(z-z_k)-(z-z_k),\quad I_\varphi=\int_{z_0}^{z}F_\varphi dz,\quad I_f=\int_{z_0}^{z}F_f dz\end{aligned} \tag{4.4}$$

However, the integrals I_φ and I_f are functions of a complex variable z which, when a circuit is described around one of the contours, may receive increments of the form $2\pi i\alpha_k$ and $2\pi i\gamma_k$, respectively; α_k and γ_k are generally complex constants (the factor 2π is introduced for convenience). Proceeding exactly as before, we have

$$I_\varphi=\sum\alpha_k\ln(z-z_k)+\varphi^*(z),\quad I_f=\sum\gamma_k\ln(z-z_k)+f^*(z) \tag{4.5}$$

where $\varphi^*(z)$ and $f^*(z)$ are single-valued functions.

Using (4.5), we write Eqs (4.4) in the form

$$\begin{aligned}
&\varphi(z) = z\sum A_k^* \ln(z - z_k) + \sum \beta_k \ln(z - z_k) + \varphi_0(z), \quad \beta_k = (\alpha_k - A_k^* z_k) \\
&f(z) = z\sum C_k^* \ln(z - z_k) + \sum \delta_k \ln(z - z_k) + f_0(z), \quad \delta_k = (\gamma_k - C_k^* z_k) \\
&\varphi_0(z) = \varphi^*(z)\sum A_k^* \ln(z - z_k), \quad f_0(z) = f^*(z) + \sum C_k^*(z - z_k)
\end{aligned} \tag{4.6}$$

where β_k and δ_k are certain complex constants.

Similar arguments establish that $\psi'(z)$ is a holomorphic function and

$$\varphi(z) = \sum B_k^* \ln(z - z_k) + \psi_0(z) \tag{4.7}$$

where $\psi_0(z)$ is a single-valued function.

The coefficients A_k^*, B_k^*, C_k^*, β_k and δ_k obey certain relations, implied by the fact that the displacements and stresses are single-valued. In the fist place, we use the relation

$$4\mu(u_1 + iu_2) = f(z) - 2[\varphi + z\overline{\varphi}'(\bar{z}) + \overline{\psi}'(\bar{z})]$$

Substituting Eqs (4.6) and (4.7) into this relation, we obtain the following expression for the increment imparted to the displacements by a circuit around the kth contour in the positive direction

$$4\mu(u_1 + iu_2)\big|_k = 2\pi i(zC_k^* + \delta_k - 2\beta_k + 2\overline{B}_k^*) \tag{4.8}$$

Obviously, a necessary condition for the displacements to be single-valued is that expression (4.8) be equal to zero, which yields two equations

$$C_k^* = 0, \quad \delta_k - 2\beta_k + 2\overline{B}_k^* = 0$$

The components of the force vector applied to the contour element ds are determined by the expressions

$$P_1 = 2f_1 h ds, \quad P_2 = 2f_2 h ds$$

It is more convenient to express the components as a combination $(P_i + iP_2)2hds$; transforming this combination in the same way as the analogous expression in the previous section, we obtain

$$(P_1 + iP_2)2hds = 2h\left(S_3 dz - \frac{\mu}{\lambda + \mu} S_1 dz + \overline{S}_2\right) \tag{4.9}$$

A necessary condition for the stresses to be single-valued is that expression (4.9) receive zero increment for any circuit around a closed contour L_k' enclosing the kth contour L_k [1]. This yields an expression for the increment of the forces (or equivalently, of the stresses) due to a circuit around the kth contour, using (4.3) and the fact that $\psi'(z)$ is holomorphic:

$$C_k^* - \frac{\mu}{\lambda + \mu} A_k^* = 0.$$

Hence it follows that $A_k^* = 0$ for any k since, as established previously, $C_k^* = 0$ for all k.

The expression for the differential of the moment M of the forces applied to an element ds of the contour L_k about the origin is

$$dM = (x_1 f_2 - x_2 f_1)2h(z, \bar{z})ds$$

Taking relations (3.1) into account, we obtain

$$dM = 2h\left\{x_1\left(\mu g dx_1 - d\frac{\partial\Phi}{\partial x_1}\right) - x_2\left(d\frac{\partial\Phi}{\partial x_2} - \mu g dx_2\right)\right\} =$$

$$= 2h\left\{\mu g(x_1 dx_1 + x_2 dx_2) - \left(x_1 d\frac{\partial\Phi}{\partial x_1} + x_2 d\frac{\partial\Phi}{\partial x_2}\right)\right\} = h\left\{\mu g d(z, \bar{z}) - \left(z d\frac{\partial\Phi}{\partial z} + \bar{z} d\frac{\partial\Phi}{\partial \bar{z}}\right)\right\}$$

This relation may be written as follows in terms of the complex potentials

$$dM = h(z,\bar{z})\left\{\left[\bar{z}\left(\frac{\mu}{\lambda+\mu}S_1 - S_3\right) - zS_2\right]dz + \left[z\left(\frac{\mu}{\lambda+\mu}S_1 - S_3\right) - \bar{z}\bar{S}_2\right]d\bar{z}\right\}$$

It is obvious that the constants $A_k^* = 0$, $C_k^* = 0$ determined above also guarantee that the moment will be single-valued. Thus, the complex functions under consideration are indeed holomorphic.

REFERENCES

1. MUSKHELISHVILI, N. I., *Some Basic Problems of the Mathematical Theory of Elasticity*. Noordhoof, Leyden, 1975.
2. TIMOSHENKO, S. P. and GOODIER, J. N., *Theory of Elasticity*. McGraw-Hill, New York, 1970.
3. LOVE, A., *A Treatise on the Mathematical Theory of Elasticity*. Cambridge University Press, Cambridge, 1928.

Translated by D.L.